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NOTE

Computation of the Ideal-MHD Continuous Spectrum in Axisymmetric Plasmas

1. INTRODUCTION

Many phenomena in laboratory plasmas, solar and stellar plasmas, and the earth's magnetosphere are dominated by a magnetic field and are described by the equations of magnetohydrodynamics (MHD for short). Spectral analysis by means of the normal-mode approach is often indispensable to obtain a deeper insight in such phenomena. In linear ideal (dissipationless) MHD, the spectrum of oscillation frequencies of an inhomogeneous plasma consists of a continuous spectrum in addition to the discrete spectrum. The continuum modes are characterized by spatial non-square integrable singularities and a non-exponential growth or decay in time. The importance of the continuous spectrum is beyond dispute: it is essential for the determination of the temporal behaviour of the plasma and for the understanding of the structure of the spectrum since it serves as the only location of possible accumulation points. Also, the knowledge of the internal structure of the continuous spectrum, and hence the location of the singularities, is a clear—and in practice often indispensable—advantage for accurately solving the MHD equations. In addition, the resonant absorption of Alfvén continuum waves is a promising scheme for plasma heating. Hence, the ideal continuous spectrum has been studied by many authors (see, e.g., [1–6] and references therein).

The ideal MHD equations that govern the linear perturbations around a magnetostatic equilibrium can be written in the form

$$\frac{\partial \rho_1}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{v}_1), \tag{1a}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + (\nabla \times \mathbf{B}_0) \times (\nabla \times \mathbf{A}_1) + (\nabla \times \nabla \times \mathbf{A}_1) \times \mathbf{B}_0, \tag{1b}$$

$$\rho_0 \frac{\partial T_1}{\partial t} = -\rho_0 \mathbf{v}_1 \cdot \nabla T_0 - (\gamma - 1)\, \rho_0 T_0 \nabla \cdot \mathbf{v}_1, \tag{1c}$$

$$\frac{\partial \mathbf{A}_1}{\partial t} = \mathbf{v}_1 \times \mathbf{B}_0, \qquad \text{where} \quad \mathbf{B}_1 = \nabla \times \mathbf{A}_1, \tag{1d}$$

where ρ is the plasma density, T is the temperature, $\mathbf{B}$ is the magnetic field, and $\mathbf{v}$ is the velocity field. The subscript 1 denotes an Eulerian perturbation and the subscript 0 indicates an equilibrium quantity. The equilibrium is assumed to be given in flux coordinates (ψ, θ, ϕ) with ψ the coordinate that labels the magnetic surfaces. The perturbed magnetic field is represented by a vector potential $\mathbf{A}_1$. The ratio of specific heats, γ, is taken to be $\frac{5}{3}$. Equations (1a)–(1d) form a system of eight *partial* differential equations for eight unknowns, viz. ρ_1, $v_{1\psi}$, $v_{1\theta}$, $v_{1\phi}$, T_1, $A_{1\psi}$, $A_{1\theta}$, and $A_{1\phi}$. With the assumption of a time dependence of the form $e^{\lambda t}$ and appropriate boundary conditions the system (1) determines a complex eigenvalue problem of the form

$$\mathbf{R} \cdot \mathbf{u} = \lambda \mathbf{S} \cdot \mathbf{u}, \tag{2}$$

with $\mathbf{u}$ a state vector that contains the perturbed quantities and λ the (complex) eigenvalue. The ideal-MHD operator is represented by matrices $\mathbf{R}$ and S, where S is a diagonal matrix and $\mathbf{R}$ contains differential operators and equilibrium quantities.

When the equilibrium quantities vary only in one spatial direction and are constant on magnetic surfaces, the perturbed quantities can be Fourier analyzed in the two ignorable spatial coordinates and each Fourier component can be studied separately since there is no interaction between different mode numbers. The continuous parts of the spectrum can then be determined in an algebraic manner as they correspond to the singularities of the coefficients of a second-order ordinary differential equation (the Hain–Lüst equation). The Alfvén continuum, for instance, is given by the dispersion relation $\omega_a(\psi) = \mathbf{k}(\psi) \cdot \mathbf{B}_0(\psi)/\sqrt{\mu \rho_0(\psi)}$, with $\mathbf{k}$ the wave vector. In equilibria with inhomogeneity in two directions, however, the equilibrium quantities also vary on magnetic surfaces and this loss of symmetry makes the determination of the continuous spectrum considerably more complicated. The continuous spectrum of static, axisymmetric, and toroidal plasmas has been derived independently by Goedbloed and Pao [1, 2]. These authors showed that in a two-dimensional plasma the continuous spectrum is determined by a reduced

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(one-dimensional non-singular) eigenvalue problem for a fourth-order system of ordinary differential equations on each flux surface. These reduced equations are obtained upon limiting the analysis to the neighbourhood of a particular flux surface $\psi = \psi_0$ and neglecting smaller terms. The coefficients of this reduced eigenvalue problem are all non-singular and, consequently, this eigenvalue problem yields a discrete set of eigenvalues on each flux surface. However, each eigenvalue of this discrete set spreads out a continuous spectrum when the magnetic surface is varied. It can be shown [1, 2] that a solution of the non-singular eigenvalue problem on $\psi = \psi_0$ corresponds to an improper solution of the original problem (2) with the same eigenvalue and with $v_{1\psi}$, $a_{1\theta}$, and $a_{1\phi}$ (and, hence, $B_{1\psi}$) diverging logarithmically as $\psi \to \psi_0$, while the other components of the state vector diverge as $(\psi - \psi_0)^{-1}$.

The numerical method presented in the next section is essentially a convenient way to obtain the equations of the reduced eigenvalue problem that determines the continuous spectrum from the set of full ideal MHD equations (1) by using the results of Pao and Goedbloed and avoiding any analytic manipulations of these equations. In Section 3, this method is demonstrated upon computing the continuous spectrum and its internal structure for a fusion-relevant tokamak plasma.

2. NUMERICAL METHOD

The numerical method to compute the ideal-MHD continuous spectrum is now described by means of a concrete example in order to demonstrate its simplicity and convenient implementation when starting from a given spectral code. The normal-mode code CASTOR (complex Alfvén spectrum for toroidal plasmas) [7, 8] is used as a starting point. CASTOR solves the resistive-MHD eigenvalue problem by means of the Galerkin procedure in combination with a finite-element discretization in the ψ-direction (normal to the flux surfaces) and a Fourier expansion in the poloidal coordinate. A combination of two cubic Hermite finite elements (for $v_{1\psi}$, $a_{1\theta}$, and $a_{1\phi}$) and two quadratic finite elements (for $v_{1\theta}$, $v_{1\phi}$, ρ_1, and $a_{1\psi}$) is used to avoid "spectral pollution" and to improve the accuracy. This leads to a general matrix eigenvalue problem of the form

$$\mathbf{A} \cdot \mathbf{a} = \lambda \mathbf{B} \cdot \mathbf{a}, \tag{3}$$

with $\mathbf{a}$ the vector of the $16 \times N_\psi \times N_m$ expansion coefficients, where N_ψ is the number of radial mesh points and N_m is the number of poloidal fourier components ($\sim e^{im\theta}$). The problem now is that, in order to determine the continuous part of the spectrum, N_ψ has to be large because the singularities have to be resolved properly. Hence, the system (3) becomes very large and expensive to solve, especially when strong mode coupling occurs and N_m has to be large too. The required radial grid points can be substantially reduced when the location of the singularities is known, namely by mesh accumulation at these flux surfaces. This, however, requires information on the internal structure of the continuous spectrum, i.e., on the ψ-profiles of the local Alfvén frequencies.

Given a solver for the general eigenvalue problem (3), such as CASTOR for vanishing resistivity, the ideal continuous spectrum can easily be obtained by prescribing the radial dependence which is known to be singular and logarithmic in nature (see the previous section). This is done by focussing on one magnetic flux surface and replacing the cubic finite elements by $\log(\varepsilon)$ and the quadratic elements by $1/\varepsilon$, with small ε, e.g., 10^{-8}. This approximates the logarithmic singularity of the radial velocity and magnetic field components, and the $(\psi - \psi_0)^{-1}$-type singularity of the other dependent variables, and introduces an ordering of the terms in the system (3) analoguous to the ordering introduced by Pao and Goedbloed to obtain a reduced eigenvalue problem on $\psi = \psi_0$. The resulting eigenvalue problem is solved with the QR-algorithm for a finite number of ψ_0's. The result is shown in the next section. The reduced eigenvalue problem solved on each flux surface is then determined by only $8 \times N_m$ equations and yields the continuum frequencies and the (regular) dependence of the continuum modes on the poloidal coordinate.

3. RESULTS AND DISCUSSION

Consider a toroidal, axisymmetric plasma with circular cross section, a small pressure ($\beta_p \approx 1\%$), an aspect ratio of 2.5, and a safety factor raising from $q_0 = 1.05$ on the magnetic axis to $q_s = 2.3$ at the plasma surface. As for CASTOR, the equilibrium is computed by means of the equilibrium solver HELENA [9] in non-orthogonal flux coordinates (ψ, θ, ϕ) with straight field lines. The density profile is chosen as

$$\rho_0(\psi) = 1 - 0.95s^2, \tag{4}$$

with $s = \sqrt{\psi/\psi_s}$. The ψ-profiles of the continuous spectra for the toroidal mode number $n = -3$ and the poloidal mode numbers $m = 2, 3, 4, 5$, and 6 are displayed in Fig. 1a. The reduced eigenvalue problem has been solved on 100 flux surfaces (100 equidistant s-values) which clearly suffices to reveal the internal structure of the ideal continuous spectrum which is up-down symmetric. The $q(\psi)$-profile is also indicated on Fig. 1a. The toroidicity of the equilibrium induces poloidal mode coupling. The numbers on the continuum branches indicate the dominant poloidal mode number m for that branch. The smooth curves near $\text{Im}(\lambda) = 0$ belong to the slow magnetosonic continuum which extends over a much smaller range because of the low equilibrium pressure. The effect of the poloidal mode

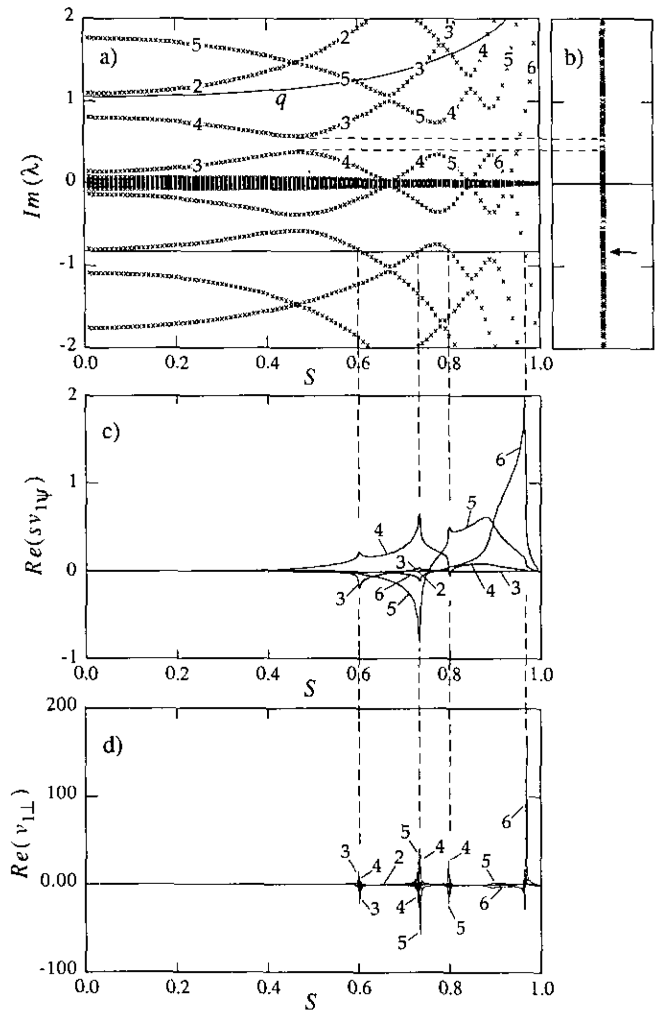

FIG. 1. (a) Structure of the ideal-MHD continuous spectrum for $n=-3$ and $m=2$–6 and a density profile given by Eq. (4). (b) Corresponding ideal-MHD spectrum as obtained with CASTOR ($N_\psi=21$, $N_m=5$). (c) Radial structure of the $sv_{1\psi}$-component of the continuum mode with $\mathrm{Im}(\lambda)=-0.82$, obtained with CASTOR ($N_\psi=251$, $N_m=5$). (d) Radial structure of the $v_{1\perp}$-component of the continuum mode with $\mathrm{Im}(\lambda)=-0.82$.

coupling on the continuous Alfvén spectrum is particularly strong on rational surfaces and neighbouring magnetic surfaces. Modes (m, n) and (m', n), with different poloidal mode numbers, couple strongly on the (rational) magnetic surface, where $q(\psi)=-(m+m')/2n$. On such rational surfaces the corresponding one-dimensional continuum frequencies are degenerate. The strong poloidal mode coupling removes these degeneracies and produces gaps in the continuous spectrum of toroidal equilibria. In Fig. 1a such gaps occur, for instance: near $s=0.47$, where $q=\frac{7}{6}$ and $m=3$ and 4 couple strongly; near $s=0.67$ ($q=\frac{4}{3}$, $m=3$ and 5); and near $s=0.78$ ($q=1.5$, $m=4$ and 5).

The total ideal-MHD spectrum in the range $-2\leqslant \mathrm{Im}(\lambda)\leqslant +2$, as obtained from CASTOR with 21 radial gridpoints and five Fourier components, is displayed in Fig. 1b. Clearly, this picture contains much less information on the continuous spectrum. Indeed, spectral codes like CASTOR can only compute the projections of the ψ-profiles shown in Fig. 1a. Owing to the finite spatial resolution, the continua show up as closely spaced discrete eigenvalues. The gap $0.40\leqslant \mathrm{Im}(\lambda)\leqslant 0.59$ at $s=0.47$ shows up here too (as indicated on the figure), but the other gaps are hardly recognizable because of overlapping continuum branches. Yet, CASTOR consumes a lot more CPU-time: the continuous spectrum, as shown in Fig. 1a, is computed in 12 s on a Cray X-MP, while the CASTOR result shown in Fig. 1b required about 50 times more CPU-time. The reason, of course, is that CASTOR not only computes the continuous part, but the complete ideal-MHD spectrum, and has to resolve the singular ψ-dependence to do so. This singular radial (ψ-) structure of the continuum modes can be determined with CASTOR by an inverse vector iteration. In Fig. 1c and d, for instance, the radial dependence of the continuum mode with $\mathrm{Im}(\lambda)=-0.82$ is displayed. The $v_{1\psi}$-component clearly has four logarithmic singularities on the four magnetic surfaces, where the local Alfvén frequency equals -0.82, while the tangential velocity component perpendicular to the magnetic field lines has $(\psi-\psi_0)^{-1}$-singularities on these surfaces. As indicated, the location of the singularities as well as the respective dominant Fourier components are in perfect agreement with the result shown in Fig. 1a: on $s=0.60$ the $m=3$-component is dominant; on $s=0.74$ the $m=5$-component is dominant; on $s=0.80$ the $m=4$-component is dominant; and on $s=0.96$ the $m=6$-component is dominant. This kind of information is very helpful and sometimes even indispensable, for instance, for computing the damping of so-called gap-modes [7, 10, 11], such as the one in the lower part of the [0.40, 0.59]-gap indicated on Fig. 1b. These gap-modes, or toroidicity-induced Alfvén eigenmodes, might be destabilized upon interaction with α-particles. However, when there are continuum branches overlapping the gaps, like the $m=6$-branch in Fig. 1a, these modes are damped upon interaction with the corresponding continuum modes, which might be sufficient to prevent the α-particle destabilization. In order to compute the finite damping in the limit as the plasma resistivity η vanishes, the nearly-singularities (for $\eta\neq 0$) have to be resolved and for this purpose, information on the structure of the continuous spectrum is of invaluable importance [7, 11].

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REFERENCES

1. J. P. Goedbloed, *Phys. Fluids* **18**, 1258 (1975).
2. Y.-P. Pao, *Nucl. Fusion* **15**, 631 (1975).
3. S. Poedts, D. Hermans, and M. Goossens, *Astron. Astrophys.* **151**, 16 (1985).
4. S. Poedts and M. Goossens, *Solar Phys.* **109**, 265 (1987).
5. S. Poedts and M. Goossens, *Astron. Astrophys.* **198**, 331 (1988).
6. S. Poedts and M. Goossens, *Solar Phys.* **133**, 218 (1991).
7. W. Kerner, S. Poedts, J. P. Goedbloed, G. T. A. Huysmans, B. Keegan, and E. Schwarz, in Proceedings 18th Fusion Conference, Berlin, 1991.
8. J. P. Goedbloed, G. T. A. Huysmans, S. Poedts, G. Halberstadt, W. Kerner, and E. Schwarz, in Proc. IAEA Technical Committee Meeting on "Advances in Simulation and Modeling of Thermonuclear Plasmas," Montréal, 1992, in press.
9. G. T. A. Huysmans, J. P. Goedbloed, and W. Kerner, Proceedings Europhysics 2nd Int. Conf. on Computational Physics, Amsterdam, 1990.
10. C. Z. Cheng, and M. S. Chance, *Phys. Fluids* **29**, 3695 (1986).
11. S. Poedts, W. Kerner, J. P. Goedbloed, B. Keegan, G. T. A. Huysmans, and E. Schwarz, *Plasma Phys. Controlled Fusion* **34**, 1397 (1992).

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